Background risk in generalized expected utility theory

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Abstract

In this paper, it is shown that, for a wide range of risk-averse generalized expected utility preferences, independent risks are complementary, contrary to the results for expected utility preferences satisfying conditions such as proper and standard risk aversion.

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1 Introduction

The response of risk-averse decision-makers with expected-utility preferences to the addition of independent background risk has been the subject of a substantial literature. Important contributions include those of Pratt and Zeckhauser (1987), Pratt (1988), Kimball (1993) and Gollier and Pratt (1996). The literature has recently been summarized and extended by Gollier (2000).

The primary result of this literature, consistent with the intuitive concept of risk aversion, is that the addition of independent risk reduces welfare: more risk is worse. A more complex question is whether aversion to a given risk should increase in the presence of an independent background risk. On the one hand, intuition based on convexity of preferences suggests that the higher the underlying level of risk, the greater the marginal cost of additional risk. On the other hand, psychological evidence of diminishing sensitivity suggests that if the level of risk is high in any case, people will not be particularly concerned about the addition of a small independent risk. Gollier and Pratt (1996) describe the property of increasing sensitivity in response to independent background risk as risk vulnerability and show that a sufficient condition for risk vulnerability is that absolute risk aversion is decreasing and convex. Standard risk aversion in the sense of Kimball (1993) and proper risk aversion in the sense of Pratt and Zeckhauser (1987) both imply risk vulnerability.

The seminal analysis of Machina (1982) established that any decision-maker with everywhere-concave local utility functions would satisfy the preference-ordering of second-order stochastic dominance and would therefore be strictly worse off as a result of the addition of independent background risk. Beyond this basic point, there has been relatively little analysis of the impact of background risk on the choices of decision-makers with generalized expected utility preferences.

The main purpose of the present paper is to analyze the impact of background risk for the case of constant risk-aversion in the sense of Safra and Segal (1998) and Quiggin and Chambers (1998). Constant risk aversion arises when neither a fixed increment to all risky outcomes nor a scalar multiplication of outcomes affects preference rankings. An important special case of constant risk aversion is that of rank-dependent preferences with linear utility, first analyzed by Yaari (1987) under the name the 'dual theory'. Analogous functional forms have also arisen in the theory of inequality measurement, in the form of generalizations of the Gini coefficient (Weymark 1981). Safra and Segal show that the combination of constant risk aversion, preference for diversification and 'zero-independence' (a generalization of the homogeneity assumption of Rubinstein, Safra and Thomson 1992 and Grant and Kajii 1995) is equivalent to the requirement that preferences must be representable by the S-Gini functional form of Donaldson and Weymark (1980).

In this note, it is shown that, for preferences displaying constant risk aversion, the premium for a given risk is always reduced by the presence of independent background risk, the opposite of the result found for expected utility with standard preferences.

2 Model

A positive random variable may be considered as a mapping from a state-space S to an outcome space $X \subseteq \Re$, with an associated measure on S given by the probability vector $(\pi_1, ..., \pi_S)$, where, by a slight abuse of notation, S is used to denote both the state-space and its cardinality.

Background risk may be represented by a Cartesian product space

$$S = S_1 \times S_2$$

and random variables of the form

$$\tilde{\mathbf{x}} = \mathbf{x} + \boldsymbol{\varepsilon}$$

where $\mathbf{x} : S_1 \times S_2 \rightarrow \Re$ is measurable with respect to S_1, $\boldsymbol{\varepsilon} : S_1 \times S_2 \rightarrow \Re$ is measurable with respect to S_2, and $E[\boldsymbol{\varepsilon}] = 0$. Note that the assumption that $E[\boldsymbol{\varepsilon}] = 0$ involves no loss of generality. Also, observe that, with this setup, addition of random variables is represented as a sum of finite dimensional vectors, with the usual vector space properties, so that, for example

$$\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} = 2\boldsymbol{\varepsilon}.$$

More generally, the space of random variables with background risk of the form $\mathbf{x} + \boldsymbol{\varepsilon}$ represents a subspace of the space $\Re^{S_1 \times S_2}$ with generic element denoted $\mathbf{y}$.

Preferences are represented by mappings of the form $V : \Re^{S_1 \times S_2} \to \Re$. It is convenient to characterize V by the associated certainty equivalent

$$e(\mathbf{y}) = inf\{c : V(c\mathbf{1}) \geq V(\mathbf{y})\}$$

where $\mathbf{1} = (1, ...1)$ is the unit vector in $\Re^{S_1 \times S_2}$. We focus on the case where V is a preference function displaying constant risk aversion in the sense of Safra and Segal (1998). Constant risk aversion requires both constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA). That is, for all $\mathbf{y}$, all δ and all $t > 0$,

$$\begin{aligned} e(\mathbf{y}+\delta\mathbf{1}) &= e(\mathbf{y})+\delta \quad \text{(CARA); and} \\ e(t\mathbf{y}) &= te(\mathbf{y}) \quad \text{(CRRA).} \end{aligned}$$

The certainty equivalent is quasi-concave (preferences are convex), if, for all $\mathbf{y}, \mathbf{y}'$ and $\lambda \in [0, 1]$

$$e(\lambda\mathbf{y}'+(1-\lambda)\mathbf{y}) \geq min(e(\mathbf{y}'),e(\mathbf{y})).$$

Preferences are risk averse if for all $\mathbf{y}$,$e(\mathbf{y}) \leq E[\mathbf{y}]$. Under expected utility theory, quasi-concavity is equivalent to risk-aversion, but this equivalence does not hold in general. For example, state-dependent preferences may be quasi-concave, but not risk-averse in the sense that $e(\mathbf{y}) \leq E[\mathbf{y}]$, where the expectation may be taken with respect to probabilities derived as in Grant and Karni (2000). Conversely, Dekel (1989) shows that preferences may be risk-averse but not quasi-concave.

With this background, we can derive:

Proposition 1 Let $\mathbf{x}$ be a random variable and let $\boldsymbol{\varepsilon}$ be an independent background risk. Then, if the certainty equivalent e is quasi-concave and displays constant risk-aversion,

$$e(\mathbf{x}+\boldsymbol{\varepsilon}) \geq e(\mathbf{x})+e(\varepsilon).$$

Proof: Let

$$\delta = e(\mathbf{x})-e(\varepsilon)$$

so

$$\begin{aligned} e(\mathbf{x}) &= e(\varepsilon + \delta 1) \quad \text{by CARA} \\ &\leq e\left(\frac{\mathbf{x}+\varepsilon + \delta 1}{2}\right) \quad \text{by quasi-concavity} \\ &= e\left(\frac{\mathbf{x}+\varepsilon}{2}\right) + \frac{\delta}{2} \quad \text{by CARA} \\ &= \frac{1}{2} e\left(\mathbf{x}+\varepsilon\right) + \frac{\delta}{2} \quad \text{by CRRA.} \end{aligned}$$

Hence, by CRRA

$$\begin{aligned} e\left(\mathbf{x}+\varepsilon\right) &\geq 2e(\mathbf{x}) - \delta \\ &= e(\mathbf{x}) + e(\varepsilon) \qquad \blacksquare \end{aligned}$$

The only class of expected-utility preferences displaying constant risk-aversion are risk-neutral preferences. However, there are a number of interesting examples of generalized expected utility preferences displaying constant risk aversion, some of which have been discussed in more detail by Safra and Segal (1998). Safra and Segal (1988, Lemma 4) show that all quasiconcave preferences display constant risk aversion if and only if they are members of the class of multiple prior expected utility maximizers with linear utility. The associated preference representation is

$$W\left(\mathbf{y}\right) = \min_{\pi \in \Pi} \sum_{s} \pi_s u\left(y_s\right)$$

where Π is a convex subset of the S-dimensional unit simplex. Gilboa and Schmeidler (1989) axiomatise this class of preferences in an Anscombe-Aumann framwork in whichan act is a mapping from the state space to the set of (finite support) lotteries defined over the outcome space. It is straightforward to modify this axiomatisation to cover the case of a one-dimensional outcome space considered here.[1]

Examples of preferences encompassed by the class of multiple prior expected utility maximizers with linear utility and quasiconcave preferences

[1] I thank a referee for pointing this out.

include: risk-averse Choquet expected uitlity with linear utility; disappointment aversion (Gul 1991) with linear utility; maximin preferences; and mean-standard deviation preferences of the form

$$e(\mathbf{x}) = E(\mathbf{x}) - k\sigma(\mathbf{x})$$

3 Concluding comments

The use of terminology such as 'standard' and 'proper' in the expected-utility literature indicates the expectation that aversion to one risk will be enhanced in the presence of another, that is, that independent risks are substitutes rather than complements. Although necessary and conditions for global substitutability between risks are quite complex, there has been little interest in the corresponding conditions for complementarity.

The analysis in this note shows that, for a wide range of risk-averse preferences derived from alternative models of choice under uncertainty, independent risks are complementary, that is, aversion to one risk will be reduced by the presence of an independent background risk. This is the opposite of the prediction for plausible forms of expected-utility preferences. The derivation of directly opposite predictions regarding comparative statics provides an opportunity for testing the relative ability of competing models to explain economic behavior. By contrast, tests of axiomatic properties such as independence have generally been confined to hypothetical choices in a laboratory setting.

These results are of particular interest with respect to inequality, where much analysis is based on inequality measures displaying the homotheticity properties referred to here as constant risk aversion. The results derived here show that the increase in inequality associated with a given source of inequality, such as differences in inherited wealth, is reduced in the presence of other, statistically independent sources of inequality, such as differences in genetic endowments.

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